§. Sparse & Carleson Families:

Let $\mathcal{D}$ be a dyadic lattice on $\mathbb{R}^n$.

DEF.: Let $0<\eta<1$. A collection $\mathcal{S}\subset\mathcal{D}$ is called **η-sparse** if for every $Q\in\mathcal{S}$ there is a measurable subset $E_Q\subset Q$ such that the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint and

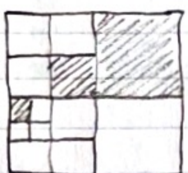
$$\boxed{|E_Q|\geq \eta|Q|,\ \forall Q\in\mathcal{S}} \Rightarrow |Q\setminus E_Q|\leq(1-\eta)|Q|.$$

DEF.: Let $\Lambda>1$. A family of cubes $\mathcal{S}\subset\mathcal{D}$ is called **Λ-Carleson** if:

$$\boxed{\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P|\leq\Lambda|Q|,\ \forall Q\in\mathcal{D}}$$

Remark: It suffices to verify the Carleson condition on all $Q\in\mathcal{S}$:

$$\boxed{\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P|\leq\Lambda|Q|,\ \forall Q\in\mathcal{S}} \Rightarrow \boxed{\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P|\leq\Lambda|Q|,\ \forall Q\in\mathcal{D}}$$



$Q\in\mathcal{D}$

Let $Q\in\mathcal{D}$. If Q contains no elements of $\mathcal{S}$, we are done (LHS is 0). If $Q\in\mathcal{S}$ we are also done. So assume $Q\notin\mathcal{S}$ but Q contains some elements of $\mathcal{S}$. Let $\boxed{ch_{\mathcal{S}}Q}$ (the "$\mathcal{S}$-children" of Q) be the maximal elements of $\mathcal{S}$ that are strictly contained in Q. Then:

$$\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P|=\sum_{Q'\in ch_{\mathcal{S}}Q}\underbrace{\sum_{\substack{P\in\mathcal{S}\\ P\subset Q'}}|P|}_{\leq\Lambda|Q'|\text{ b/c }Q'\in\mathcal{S}}\leq\Lambda\sum_{Q'\in ch_{\mathcal{S}}Q}|Q'|\leq\Lambda|Q|$$ b/c the $Q'\in ch_{\mathcal{S}}Q$ are disjoint. ∎

Remarkable Property: The sparseness & Carleson conditions are equivalent:

$\boxed{\eta\text{-sparse}}\Rightarrow\boxed{\frac{1}{\eta}\text{-Carleson}}$ (easy)

Let $Q\in\mathcal{S}$. Then:

$$\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}|P|\leq\sum_{\substack{P\in\mathcal{S}\\ P\subset Q}}\frac{1}{\eta}|E_P|=\frac{1}{\eta}\Big|\underbrace{\bigcup_{\substack{P\in\mathcal{S}\\ P\subset Q}}E_P}_{\text{b/c the sets }E_P\text{ are disjoint}}\Big|\leq\frac{1}{\eta}|Q|.$$

$\boxed{\Lambda\text{-Carleson}}\Rightarrow\boxed{\frac{1}{\Lambda}\text{-sparse}}$ (difficult).

$$\hookrightarrow\ |Q|\leq\frac{1}{\eta}|E_Q|=\Lambda|E_Q|$$

Remark: The sparseness property is something that can be readily used when working w/systems of cubes that are already known to be sparse, while the Carleson property is something that can be easily verified in many cases where the sparseness condition is not obvious at all.

For example, it is easy to see that:

The union of N Carleson systems w/constants $\Lambda_1, \dots, \Lambda_N$ is a Carleson system w/constant $\Lambda_1 + \dots + \Lambda_N$.

while to see directly that

The union of N sparse collections w/constants $\eta_1, \dots, \eta_N$ is a sparse system w/constant $\left(\frac{1}{\eta_1} + \dots + \frac{1}{\eta_N}\right)^{-1}$.

is next to impossible.

Check: Let $\mathcal{S}_1, \dots, \mathcal{S}_N$ be Carleson systems s.t. each $\mathcal{S}_j$ is Λ_j-Carleson. Let $\mathcal{S} := \bigcup_{j=1}^{N} \mathcal{S}_j$.
Then for any $Q \in \mathcal{S}$:

$$\sum_{\substack{P \in \mathcal{S} \\ P \subset Q}} |P| \leq \sum_{j=1}^{N} \underbrace{\sum_{\substack{P \in \mathcal{S}_j \\ P \subset Q}} |P|}_{\leq \Lambda_j |Q|} \leq \sum_{j=1}^{N} \Lambda_j |Q| = (\Lambda_1 + \dots + \Lambda_N)|Q| \Rightarrow \mathcal{S} \text{ is } (\Lambda_1 + \dots + \Lambda_N)\text{-Carl.}$$

(regardless if $Q \in \mathcal{S}_j$ or not! Remember this holds for all $Q \in \mathcal{D}$).

$\Rightarrow$ The equivalent statement: $\{\mathcal{S}_j\}_{j=1}^{N}$ are each η_j-sparse $\Rightarrow$ each is $\frac{1}{\eta_j}$-Carleson
$\Rightarrow$ their union $\mathcal{S}$ is $\left(\frac{1}{\eta_1} + \dots + \frac{1}{\eta_N}\right)$-Carleson $\Rightarrow \mathcal{S}$ is $\left(\frac{1}{\eta_1} + \dots + \frac{1}{\eta_N}\right)^{-1}$-sparse. $\blacksquare$

Remark: The S-children definition (a special case of sparse): (appears frequently in practice).

Suppose a family $S \subset \mathcal{D}$ has the property:

$$\boxed{\sum_{P \in ch_S(Q)} |P| \leq \alpha |Q|, \ \forall Q \in S}$$

for some $\alpha \in (0,1)$. Then S is $(1-\alpha)$-sparse (and therefore $\frac{1}{1-\alpha}$-Carleson).

Pf.: Let the sets $\boxed{E_Q := Q \setminus \bigcup_{P \in ch_S Q} P}$ for every $Q \in S$. Then clearly $\{E_Q\}_{Q \in S}$ are disjoint, and:

$$|E_Q| = \Big|Q \setminus \bigcup_{P \in ch_S Q} P\Big| = |Q| - \Big|\bigcup_{P \in ch_S Q} P\Big| = |Q| - \sum_{P \in ch_S Q} |P| \geq |Q| - \alpha |Q| = (1-\alpha)|Q|$$

(by disjointness of $P \in ch_S Q$)

$\Rightarrow |E_Q| \geq (1-\alpha)|Q| \Rightarrow S$ is $(1-\alpha)$-sparse.

We can also prove directly that it is a Carleson collection: for some $Q \in S$:

$$\sum_{\substack{P \in S \\ P \subset Q}} |P| = |Q| + \sum_{P \in ch_S Q} \sum_{\substack{P' \in S \\ P' \subset P}} |P'| = |Q| + \sum_{P \in ch_S Q} |P| + \sum_{P \in ch_S Q} \sum_{P' \in ch_S P} |P'| + \dots$$

$$\leq |Q| + \alpha|Q| + \underbrace{\sum_{P \in ch_S Q} \alpha |P|}_{\leq \alpha^2 |Q|} + \dots$$

$$\leq |Q| (1 + \alpha + \alpha^2 + \dots) = |Q| \frac{1}{1-\alpha} \text{ b/c } \alpha \in (0,1) \Rightarrow S \text{ is } \frac{1}{1-\alpha}\text{-Carleson}.$$

∎

§ Sparse Operators & The Linear A_2 Bound:

Let $\mathcal{D}$ be a dyadic grid on $\mathbb{R}^n$ and $\mathcal{S}\subset\mathcal{D}$ be an η-sparse collection $(0<\eta<1)$, i.e.

$\forall Q\in\mathcal{S}$, $\exists$ pairwise disjoint measurable subsets $E_Q\subset Q$ s.t. $|E_Q|\geq\eta|Q|$.

Given $\mathcal{S}$, define the sparse operator associated with $\mathcal{S}$:

$$\boxed{\mathcal{A}_{\mathcal{S}}f(x):=\sum_{Q\in\mathcal{S}}\langle f\rangle_Q \mathbb{1}_Q(x)}$$

→ Look at bounding $\mathcal{A}_{\mathcal{S}}: L^2(w)\to L^2(w)$, where $w\in A_2$:

→ The main tool used is the weighted dyadic maximal function: $\boxed{M_{\mathcal{D}}^{\mu}f(x):=\sup_{Q\ni x}\mathbb{E}_Q^{\mu}|f|}$

Recall that: Given a locally finite measure μ on $\mathbb{R}^n$:

$\|M_{\mathcal{D}}^{\mu}: L^p(\mu)\to L^p(\mu)\|\leq p'$, $\forall 1<p<\infty$.

→ Recall also the duality $(L^2(w))^*\simeq L^2(w^{-1})$, with $\Lambda f\mapsto(f,g)$, $g\in L^2(w^{-1})$, $\forall\Lambda\in(L^2(w))^*$, $f\in L^2(w)$.

In light of this, we look to bound:

$$\|\mathcal{A}_{\mathcal{S}}f\|_{L^2(w)}=\sup_{\substack{g\in L^2(w^{-1})\\ \|g\|_{L^2(w^{-1})}\leq 1}}|(\mathcal{A}_{\mathcal{S}}f,g)|\leq C\|f\|_{L^2(w)}$$

→ So let $f\in L^2(w)$ and $g\in L^2(w^{-1})$:

$$\langle|f|\rangle_Q=\frac{1}{|Q|}\int_Q|f|\,dx=\frac{1}{|Q|}\int_Q|f|w\,w^{-1}=\mathbb{E}_Q^{w^{-1}}(|f|w)\langle w^{-1}\rangle_Q$$

$$\langle|g|\rangle_Q=\mathbb{E}_Q^{w}(|g|w^{-1})\langle w\rangle_Q$$

$$|(\mathcal{A}_{\mathcal{S}}f,g)|=\left|\sum_{Q\in\mathcal{S}}\langle f\rangle_Q\langle g\rangle_Q|Q|\right|\leq\sum_{Q\in\mathcal{S}}\langle|f|\rangle_Q\langle|g|\rangle_Q|Q|$$

$$=\sum_{Q\in\mathcal{S}}\underbrace{\langle w^{-1}\rangle_Q\langle w\rangle_Q}_{\leq[w]_{A_2}}\mathbb{E}_Q^{w^{-1}}(|f|w)\,\mathbb{E}_Q^{w}(|g|w^{-1})\underbrace{|Q|}_{\leq\frac{1}{\eta}|E_Q|}$$

$$\leq\frac{1}{\eta}[w]_{A_2}\sum_{Q\in\mathcal{S}}\mathbb{E}_Q^{w^{-1}}(|f|w)\,\mathbb{E}_Q^{w}(|g|w^{-1})|E_Q|$$

$$=\frac{1}{\eta}[w]_{A_2}\sum_{Q\in\mathcal{S}}\int_{E_Q}\left(\mathbb{E}_Q^{w^{-1}}(|f|w)\mathbb{1}_Q(x)\right)\left(\mathbb{E}_Q^{w}(|g|w^{-1})\mathbb{1}_Q(x)\right)dx$$

$$\leq\frac{1}{\eta}[w]_{A_2}\sum_{Q\in\mathcal{S}}\int_{E_Q}M_{\mathcal{D}}^{w^{-1}}(|f|w)(x)\cdot M_{\mathcal{D}}^{w}(|g|w^{-1})(x)\,dx$$

↳ mutually disjoint!

$$\leq\frac{1}{\eta}[w]_{A_2}\int_{\mathbb{R}^n}M_{\mathcal{D}}^{w^{-1}}(|f|w)(x)\,M_{\mathcal{D}}^{w}(|g|w^{-1})(x)\,dx$$

$$\leq\frac{1}{\eta}[w]_{A_2}\underbrace{\|M_{\mathcal{D}}^{w^{-1}}(|f|w)\|_{L^2(w^{-1})}}_{\substack{\leq 2\|fw\|_{L^2(w^{-1})}\\ =2\|f\|_{L^2(w)}}}\underbrace{\|M_{\mathcal{D}}^{w}(|g|w^{-1})\|_{L^2(w)}}_{\substack{\leq 2\|gw^{-1}\|_{L^2(w)}\\ =2\|g\|_{L^2(w^{-1})}}}$$

$$\Rightarrow|(\mathcal{A}_{\mathcal{S}}f,g)|\leq\frac{4}{\eta}[w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}$$

$$\boxed{\|\mathcal{A}_{\mathcal{S}}: L^2(w)\to L^2(w)\|\leq\frac{4}{\eta}[w]_{A_2}\quad\forall w\in A_2}$$

[Cruz-Uribe, Martell, Pérez 2010]

John-Nirenberg Proof: The statement is scale-invariant, so we only need to prove:

$$\boxed{|\{x\in Q_0 : |b(x)-\langle b\rangle_{Q_0}|>\alpha\}| \leq C|Q_0|e^{-A\alpha}}$$

for every cube Q_0 in $\mathbb{R}^n$, $\alpha>0$, and $b\in BMO(\mathbb{R}^n)$ with $\|b\|_{BMO}=1$.

We let $\gamma>1$ be a constant to be determined later.

Stage (0): $\mathcal{S}_{(0)} := \{Q_0\}$

Stage (1): $\mathcal{S}_{(1)} := ch_{\mathcal{S}}(Q_0) :=$ Maximal subcubes S of Q_0 that satisfy:

$$\boxed{\frac{1}{|S|}\int_S |b(x)-\langle b\rangle_{Q_0}|\,dx > \gamma} \quad (S1)$$

Note: Q_0 itself does not satisfy (S1): $\frac{1}{|Q_0|}\int_{Q_0}|b(x)-\langle b\rangle_{Q_0}|\,dx \leq \|b\|_{BMO} = 1 < \gamma$.

Properties of cubes in $\mathcal{S}_{(1)}$:

→ **A(1)**: Each $S\in\mathcal{S}_{(1)}$ is strictly contained in Q_0 (the "$\mathcal{S}$-children" of Q_0).

→ **B(1)**: $\boxed{\gamma < \frac{1}{|S|}\int_S |b(x)-\langle b\rangle_{Q_0}|\,dx \leq 2^n\gamma}$ $\forall S\in\mathcal{S}_{(1)}$

Usual trick: the dyadic parent $\hat{S}$ of S was not selected $\Rightarrow \frac{1}{|\hat{S}|}\int_{\hat{S}}|b(x)-\langle b\rangle_{Q_0}|\,dx \leq \gamma$

$$\Rightarrow \frac{1}{|S|}\int_S |b(x)-\langle b\rangle_{Q_0}|\,dx \leq \frac{2^n}{|\hat{S}|}\int_{\hat{S}}|b(x)-\langle b\rangle_{Q_0}|\,dx \leq 2^n\gamma.$$

→ **C(1)**: $\boxed{|\langle b\rangle_S - \langle b\rangle_{Q_0}| \leq 2^n\gamma}$ $\forall S\in\mathcal{S}_{(1)}$ $\quad |\langle b\rangle_S - \langle b\rangle_{Q_0}| \leq \frac{1}{|S|}\int_S |b-\langle b\rangle_{Q_0}| \leq 2^n\gamma$

→ **D(1)**: $\boxed{\sum_{S\in\mathcal{S}_{(1)}}|S| \leq \frac{1}{\gamma}|Q_0|}$ **The Sparseness condition!**

$$B(1): \sum_{S\in\mathcal{S}_{(1)}}|S| < \frac{1}{\gamma}\underbrace{\sum_{S\in\mathcal{S}_{(1)}}}_{\hookrightarrow \text{disjoint!}}\int_S |b(x)-\langle b\rangle_{Q_0}|\,dx \leq \frac{1}{\gamma}\int_{Q_0}|b-\langle b\rangle_{Q_0}| \leq \frac{1}{\gamma}|Q_0|\underbrace{\|b\|_{BMO}}_{1}$$

→ **E(1)**: $\boxed{|b(x)-\langle b\rangle_{Q_0}| \leq \gamma \text{ a.e. on } Q_0\setminus\bigcup_{S\in\mathcal{S}_{(1)}} S}$

$x\in Q_0\setminus\bigcup_{S\in\mathcal{S}_{(1)}} S \Rightarrow \exists$ sequence of cubes R_K containing x, whose diameters shrink to 0, and which were not selected

$$\underset{LDT}{\Rightarrow} |b(x)-\langle b\rangle_{Q_0}| = \lim_{K\to\infty}\frac{1}{|R_K|}\int_{R_K}|b(x)-\langle b\rangle_{Q_0}|\,dx \leq \gamma, \text{ a.a. } x\in Q_0\setminus\bigcup_{S\in\mathcal{S}_{(1)}} S.$$

Stage (2): For every $S_1\in\mathcal{S}_{(1)}$, let $ch_{\mathcal{S}}(S_1) :=$ Maximal subcubes S of S_1 that satisfy:

$$\boxed{\frac{1}{|S|}\int_S |b(x)-\langle b\rangle_{S_1}|\,dx > \gamma} \quad (S2)$$

Let $\boxed{\mathcal{S}_{(2)} := \bigcup_{S_1\in\mathcal{S}_1} ch_{\mathcal{S}}(S_1)}$.

Properties:

→ **A(2)**: Each $S\in\mathcal{S}_{(2)}$ is contained (strictly) in a unique $S_1\in\mathcal{S}_{(1)}$.

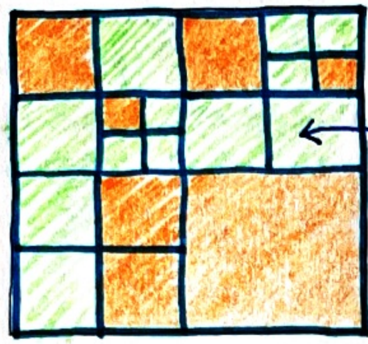
→ **B(2)**: $\boxed{\gamma < \frac{1}{|S|}\int_S |b(x)-\langle b\rangle_{S_1}|\,dx \leq 2^n\gamma}$ $\forall S\in ch_{\mathcal{S}}(S_1)$, $S_1\in\mathcal{S}_{(1)}$

→ **C(2)**: $\boxed{|\langle b\rangle_S - \langle b\rangle_{S_1}| \leq 2^n\gamma}$ $\forall S\in ch_{\mathcal{S}}(S_1)$, $S_1\in\mathcal{S}_1$

→ **D(2)**: $\boxed{\sum_{S\in\mathcal{S}_{(2)}}|S| \leq \frac{1}{\gamma}\sum_{S_1\in\mathcal{S}_{(1)}}|S_1| \leq \frac{1}{\gamma^2}|Q_0|}$

→ **E(2)**: $\boxed{|b(x)-\langle b\rangle_{S_1}| \leq \gamma \text{ a.e. on } S_1\setminus\bigcup_{S\in ch_{\mathcal{S}}(S_1)} S}$

Look more carefully at the consequences of (E1) & (E2):


Q_0

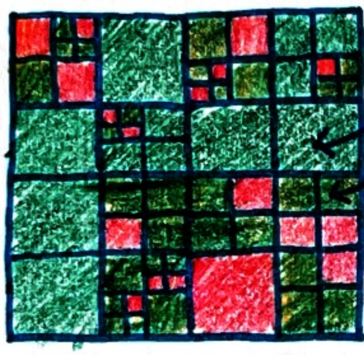
Stage (1): $\mathcal{S}_{(1)}$

On what is left of Q_0 after selecting $\mathcal{S}_{(1)}$:

$$\boxed{|b(x) - \langle b\rangle_{Q_0}| \leq \gamma} \qquad \text{(E1)}$$

(Recall: we want to bound the size of the subset of Q_0 where $|b(x) - \langle b\rangle_{Q_0}|$ is <u>greater</u> than smth.)

"Size" of $\mathcal{S}_{(1)}$: $|\cup \mathcal{S}_{(1)}| \leq \frac{1}{\gamma}|Q_0|$


Q_0

Stage (2): $\mathcal{S}_{(2)}$

On what is left of Q_0 after selecting $\mathcal{S}_{(2)}$:

$$\boxed{|b(x) - \langle b\rangle_{Q_0}| \leq 2\cdot 2^n\gamma} \text{ on } Q_0 \setminus (\cup \mathcal{S}_{(2)}) \text{ a.e.} \qquad (*)$$

=> The size of the set where $|b(x) - \langle b\rangle_{Q_0}| > 2\cdot 2^n \gamma$ would be contained in

$$|\cup \mathcal{S}_{(2)}| \leq \frac{1}{\gamma^2}|Q_0| \qquad \text{(even smaller!)}$$

$(*)$: For every $S_1 \in \mathcal{S}_{(1)}$: $|\langle b\rangle_{S_1} - \langle b\rangle_{Q_0}| \leq 2^n \gamma$ (C1)

$|b(x) - \langle b\rangle_{S_1}| \leq \gamma$ a.e. on $S_1 \setminus (\cup ch_{\mathcal{S}}(S_1))$ (E2)

=> on $S_1 \setminus (\cup ch_{\mathcal{S}}(S_1))$: $|b(x) - \langle b\rangle_{Q_0}| \leq |b(x) - \langle b\rangle_{S_1}| + |\langle b\rangle_{S_1} - \langle b\rangle_{Q_0}|$

$\leq \gamma + 2^n\gamma \leq \underline{2\cdot 2^n \gamma}$

=> on $Q_0 \setminus (\cup \mathcal{S}_{(1)})$: $|b - \langle b\rangle_{Q_0}| \leq \gamma \leq 2\cdot 2^n \gamma$

& on $(\cup \mathcal{S}_1) \setminus (\cup \mathcal{S}_{(2)})$: $|b - \langle b\rangle_{Q_0}| \leq 2\cdot 2^n \gamma$.

Stage(K): For every $S_{K-1} \in \mathcal{S}_{(K-1)}$ let $ch_{\mathcal{S}}(S_{K-1}) :=$ Maximal subcubes S of S_{K-1} such that $\boxed{\frac{1}{|S|}\int_S |b(x) - \langle b \rangle_{S_{K-1}}| dx > \gamma}$ (SK)

Let $\boxed{\mathcal{S}_{(K)} := \bigcup_{S_{K-1} \in \mathcal{S}_{(K-1)}} ch_{\mathcal{S}}(S_{K-1})}$

→ A(K): Each $S \in \mathcal{S}_{(K)}$ is contained in a unique $S_{K-1} \in \mathcal{S}_{(K-1)}$.

→ B(K): $\gamma < \frac{1}{|S|}\int_S |b(x) - \langle b \rangle_{S_{K-1}}| dx \leq 2^n \gamma \quad \forall S \in ch_{\mathcal{S}}(S_{K-1}), S_{K-1} \in \mathcal{S}_{(K-1)}$

→ C(K): $|\langle b \rangle_S - \langle b \rangle_{S_{K-1}}| \leq 2^n \gamma \quad \forall S \in ch_{\mathcal{S}}(S_{K-1}), S_{K-1} \in \mathcal{S}_{(K-1)}$

→ D(K): $\sum_{S \in \mathcal{S}_{(K)}} |S| \leq \frac{1}{\gamma^K} |Q_0|$

→ E(K): $|b(x) - \langle b \rangle_{S_{K-1}}| \leq \gamma$ a.e. on $(S_{K-1} \setminus \bigcup ch_{\mathcal{S}}(S_{K-1})) \Rightarrow \boxed{|b(x) - \langle b \rangle_{Q_0}| \leq K \cdot 2^n \gamma \text{ a.e. on } Q_0 \setminus (\bigcup \mathcal{S}_{(K)})}$

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a.e. inclusion:

$$\{x \in Q_0 : |b(x) - \langle b \rangle_{Q_0}| > K \cdot 2^n \gamma\} \subseteq (\bigcup \mathcal{S}_{(K)})$$

Need to show:

$$|\{x \in Q_0 : |b(x) - \langle b \rangle_{Q_0}| > \alpha\}| \leq C|Q_0| e^{-A\alpha}$$ for some given $\alpha > 0$.

Suppose $K \cdot 2^n \gamma < \alpha \leq (K+1) \cdot 2^n \gamma$

$$\Rightarrow |\{x \in Q_0 : |b(x) - \langle b \rangle_{Q_0}| > \alpha\}| \leq |\{x \in Q_0 : |b(x) - \langle b \rangle_{Q_0}| > K \cdot 2^n \gamma\}|$$

$$\leq |\bigcup \mathcal{S}_{(K)}| = \sum_{S \in \mathcal{S}_{(K)}} |S| \leq \frac{1}{\gamma^K} |Q_0| = |Q_0| e^{-K \log \gamma}$$

$\frac{\alpha}{2^n \gamma} \leq K+1$

$-K \leq 1 - \frac{\alpha}{2^n \gamma}$

$$\leq |Q_0| e^{\left(1 - \frac{\alpha}{2^n \gamma}\right) \log \gamma}$$

$$= |Q_0| \gamma e^{-\alpha \left(\frac{\log \gamma}{2^n \gamma}\right)}$$

If, on the other hand, $\alpha \leq 2^n \gamma$:

$$|\{x \in Q_0 : |b(x) - \langle b \rangle_{Q_0}| > \alpha\}| \leq |Q_0| \leq |Q_0| e^{(2^n \gamma - \alpha)} = |Q_0| \underline{\underline{e^{2^n \gamma}}} \cdot e^{-\alpha}$$

$\Rightarrow$ We proved the inequality for $A = \frac{\log \gamma}{2^n \gamma}$ & $C = e^{2^n \gamma}$

$\Rightarrow$ since $\gamma > 1$ was arbitrary, take $\gamma = e$ (maximizes A) $\Rightarrow A = \frac{1}{2^n e}$; $C = e^{2^n e}$

$1 \boxed{?} \frac{\log \gamma}{2^n \gamma}$ $\quad e^{2^n \gamma} \boxed{?} \gamma$

$2^n \gamma \boxed{?} \log \gamma$